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Fully Time-Domain Simulation of Multiconductor Transmission Line Systems

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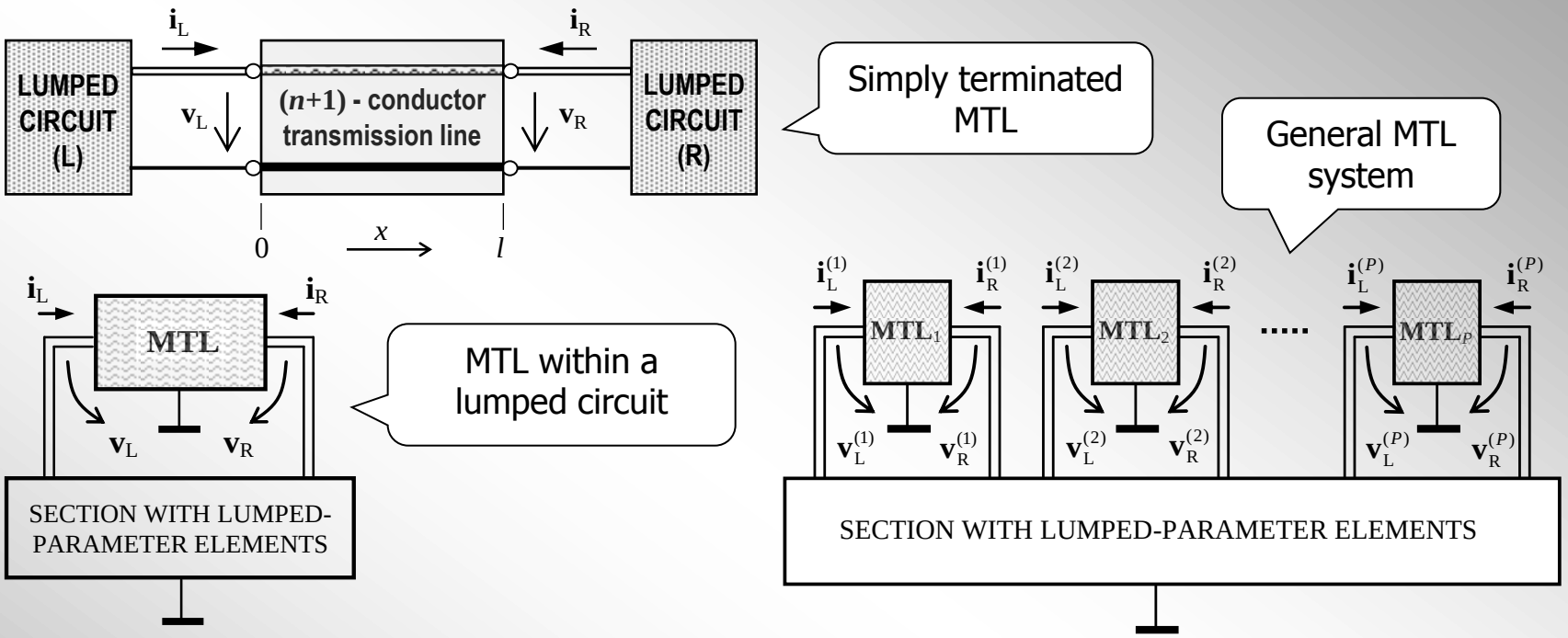
Presentation schedule

- Introduction
- Implicit Wendroff formula
- MTL boundary conditions incorporation
  - Simply terminated MTL (Thévenin equivalents)
  - MTL within a lumped circuit (MNA formulation)
  - General MTL systems (MNA, Euler method)
- Experimental error analysis
- Examples of MTL simulation
- CPU time evaluation
- Conclusion

Introduction



❖ **MTL telegraphic equations**

$$-\frac{\partial \mathbf{v}(t,x)}{\partial x} = \mathbf{R}_0(x)\mathbf{i}(t,x) + \mathbf{L}_0(x)\frac{\partial \mathbf{i}(t,x)}{\partial t}, \quad -\frac{\partial \mathbf{i}(t,x)}{\partial x} = \mathbf{G}_0(x)\mathbf{v}(t,x) + \mathbf{C}_0(x)\frac{\partial \mathbf{v}(t,x)}{\partial t}$$

$\mathbf{R}_0(x)$, $\mathbf{L}_0(x)$, $\mathbf{G}_0(x)$, $\mathbf{C}_0(x)$ – nonuniform MTL's $n \times n$ per-unit-length matrices

$\mathbf{v}(t,x)$, $\mathbf{i}(t,x)$ – $n \times 1$ column vectors of voltages and currents of n active wires

Implicit Wendroff formula

- Voltage and current vectors and their derivatives are replaced by

$$\left.\frac{\partial \mathbf{u}(t,x)}{\partial t}\right|_{j,k} \doteq \frac{1}{2}\left(\frac{\mathbf{u}_k^j - \mathbf{u}_k^{j-1}}{\Delta t} + \frac{\mathbf{u}_{k+1}^j - \mathbf{u}_{k+1}^{j-1}}{\Delta t}\right), \quad \left.\frac{\partial \mathbf{u}(t,x)}{\partial x}\right|_{j,k} \doteq \frac{1}{2}\left(\frac{\mathbf{u}_{k+1}^j - \mathbf{u}_k^j}{\Delta x} + \frac{\mathbf{u}_{k+1}^{j-1} - \mathbf{u}_k^{j-1}}{\Delta x}\right)$$

$$\left.\mathbf{u}(t,x)\right|_{j,k} \doteq \left(\mathbf{u}_{k+1}^j + \mathbf{u}_k^j + \mathbf{u}_{k+1}^{j-1} + \mathbf{u}_k^{j-1}\right)/4$$

- Equations expressed for (k+1)-th section and j-th time instance

$$\mathbf{v}_k^j - \mathbf{v}_{k+1}^j + \mathbf{A}_{vk}\mathbf{i}_k^j + \mathbf{A}_{vk}\mathbf{i}_{k+1}^j = -\mathbf{v}_k^{j-1} + \mathbf{v}_{k+1}^{j-1} + \mathbf{B}_{vk}\mathbf{i}_k^{j-1} + \mathbf{B}_{vk}\mathbf{i}_{k+1}^{j-1}$$

$$\mathbf{i}_k^j - \mathbf{i}_{k+1}^j + \mathbf{A}_{ik}\mathbf{v}_k^j + \mathbf{A}_{ik}\mathbf{v}_{k+1}^j = -\mathbf{i}_k^{j-1} + \mathbf{i}_{k+1}^{j-1} + \mathbf{B}_{ik}\mathbf{v}_k^{j-1} + \mathbf{B}_{ik}\mathbf{v}_{k+1}^{j-1}$$

with

$$\mathbf{A}_{vk} = -\left(\mathbf{R}_{0k}/2 + \mathbf{L}_{0k}/\Delta t\right)\Delta x \quad , \quad \mathbf{B}_{vk} = \left(\mathbf{R}_{0k}/2 - \mathbf{L}_{0k}/\Delta t\right)\Delta x$$

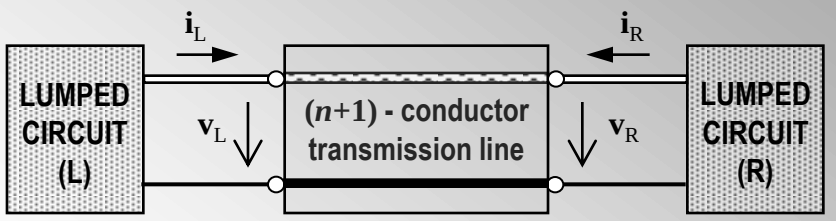
$$\mathbf{A}_{ik} = -\left(\mathbf{G}_{0k}/2 + \mathbf{C}_{0k}/\Delta t\right)\Delta x \quad , \quad \mathbf{B}_{ik} = \left(\mathbf{G}_{0k}/2 - \mathbf{C}_{0k}/\Delta t\right)\Delta x$$

where $\mathbf{R}_{0k} = \mathbf{R}_0(\xi_k)$, $\mathbf{L}_{0k} = \mathbf{L}_0(\xi_k)$, $\mathbf{G}_{0k} = \mathbf{G}_0(\xi_k)$, $\mathbf{C}_{0k} = \mathbf{C}_0(\xi_k)$, with $\xi_k \in (x_k, x_k+1)$

Simply terminated MTL

- **Matrix recursive formulation**

$$\mathbf{A}\mathbf{x}^{j} = \mathbf{B}\mathbf{x}^{j-1} + \mathbf{D}^{j}$$



Boundary conditions

$$\mathbf{x}^{j} = \left[\mathbf{v}^{jT}, \mathbf{i}^{jT}\right]^{T} \quad \text{with} \quad \mathbf{v}^{j} = \left[\mathbf{v}_1^{jT}, \mathbf{v}_2^{jT}, \ldots, \mathbf{v}_{K+1}^{jT}\right]^{T}, \quad \mathbf{i}^{j} = \left[\mathbf{i}_1^{jT}, \mathbf{i}_2^{jT}, \ldots, \mathbf{i}_{K+1}^{jT}\right]^{T}$$

- Equation internal structure (MTL divided on $K = 3$ sections)

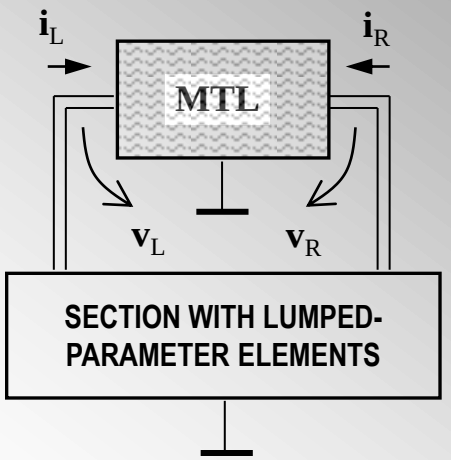
$$\left[\begin{array}{cccc|cccc}
\mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{v1} & \mathbf{A}_{v1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{v2} & \mathbf{A}_{v2} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{v3} & \mathbf{A}_{v3} \\
\hline
\mathbf{A}_{i1} & \mathbf{A}_{i1} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{A}_{i2} & \mathbf{A}_{i2} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{A}_{i3} & \mathbf{A}_{i3} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} \\
\hline
\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_{iL} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{R}_{iR}
\end{array}\right] \cdot \left[\begin{array}{c}
\mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \hline \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \\ \mathbf{i}_4
\end{array}\right]^{j} = \left[\begin{array}{cccc|cccc}
-\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{v1} & \mathbf{B}_{v1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{v2} & \mathbf{B}_{v2} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{v3} & \mathbf{B}_{v3} \\
\hline
\mathbf{B}_{i1} & \mathbf{B}_{i1} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{B}_{i2} & \mathbf{B}_{i2} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{B}_{i3} & \mathbf{B}_{i3} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right] \cdot \left[\begin{array}{c}
\mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \\ \hline \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \\ \mathbf{i}_4
\end{array}\right]^{j-1} + \left[\begin{array}{c}
\mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \hline \mathbf{v}_{iL} \\ \mathbf{v}_{iR}
\end{array}\right]^{j}$$

Boundary conditions via generalized Thévenin equivalents

MTL within a lumped circuit (1)

- **Modified nodal analysis** description

$$\mathbf{C}\frac{d\mathbf{v}_{\mathrm{N}}(t)}{dt}+\mathbf{G}\mathbf{v}_{\mathrm{N}}(t)+\mathbf{S}_{\mathrm{L}}\mathbf{i}_{\mathrm{L}}(t)+\mathbf{S}_{\mathrm{R}}\mathbf{i}_{\mathrm{R}}(t)=\mathbf{i}_{\mathrm{N}}(t)$$



- **MTL boundary conditions**

$$\mathbf{v}_{\mathrm{L}}(t)=\mathbf{S}_{\mathrm{L}}^{\mathrm{T}}\mathbf{v}_{\mathrm{N}}(t),\quad \mathbf{v}_{\mathrm{R}}(t)=\mathbf{S}_{\mathrm{R}}^{\mathrm{T}}\mathbf{v}_{\mathrm{N}}(t)$$

- **Matrix recursive formulation** $\mathbf{A}\mathbf{x}^{j}=\mathbf{B}\mathbf{x}^{j-1}+\mathbf{D}^{j}$

$$\mathbf{x}^{j}=\left[\mathbf{v}^{j\mathrm{T}},\mathbf{i}^{j\mathrm{T}},\mathbf{v}_{\mathrm{N}}^{j\mathrm{T}}\right]^{\mathrm{T}}\quad\text{with}\quad \mathbf{v}^{j}=\left[\mathbf{v}_{1}^{j\mathrm{T}},\mathbf{v}_{2}^{j\mathrm{T}},\ldots,\mathbf{v}_{K+1}^{j\mathrm{T}}\right]^{\mathrm{T}}$$

$$\mathbf{i}^{j}=\left[\mathbf{i}_{1}^{j\mathrm{T}},\mathbf{i}_{2}^{j\mathrm{T}},\ldots,\mathbf{i}_{K+1}^{j\mathrm{T}}\right]^{\mathrm{T}}$$

Wendroff method

$$\mathbf{H}=\mathbf{G}+\frac{\mathbf{C}}{\Delta t}\qquad \mathbf{F}=\frac{\mathbf{C}}{\Delta t}$$

$$\mathbf{S}_{\mathrm{L}}\mathbf{i}_{\mathrm{L}}^{j}+\mathbf{S}_{\mathrm{R}}\mathbf{i}_{\mathrm{R}}^{j}+\mathbf{H}\mathbf{v}_{\mathrm{N}}^{j}=\mathbf{F}\mathbf{v}_{\mathrm{N}}^{j-1}+\mathbf{i}_{\mathrm{N}}^{j}$$

$$\mathbf{v}_{\mathrm{L}}^{j}-\mathbf{S}_{\mathrm{L}}^{\mathrm{T}}\mathbf{v}_{\mathrm{N}}^{j}=\mathbf{0}$$

$$\mathbf{v}_{\mathrm{R}}^{j}-\mathbf{S}_{\mathrm{R}}^{\mathrm{T}}\mathbf{v}_{\mathrm{N}}^{j}=\mathbf{0}$$

MNA equations via implicit Euler method

MTL within a lumped circuit (2)

$$\mathbf{A}\mathbf{x}^{j} = \mathbf{B}\mathbf{x}^{j-1} + \mathbf{D}^{j}$$

$$\mathbf{A} = \left[\begin{array}{cc|c} \mathbf{I}_{\pm} & \mathbf{A}_{\mathrm{v}} & \mathbf{0} \\ \mathbf{A}_{\mathrm{i}} & \mathbf{I}_{\pm} & \mathbf{0} \\ \hline \mathbf{I}_{2} & \mathbf{0} & \mathbf{S}_{\mathrm{c}} \\ \mathbf{0} & \mathbf{S}_{\mathrm{r}} & \mathbf{H} \end{array}\right], \quad \mathbf{B} = \left[\begin{array}{cc|c} \mathbf{I}_{\mp} & \mathbf{B}_{\mathrm{v}} & \mathbf{0} \\ \mathbf{B}_{\mathrm{i}} & \mathbf{I}_{\mp} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{array}\right], \quad \mathbf{D} = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{i}_{\mathrm{N}} \end{array}\right]$$

- Equation internal structure (MTL divided on $K = 3$ sections)

$$\left[\begin{array}{cccc|cccc|c} \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\mathrm{v1}} & \mathbf{A}_{\mathrm{v1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\mathrm{v2}} & \mathbf{A}_{\mathrm{v2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\mathrm{v3}} & \mathbf{A}_{\mathrm{v3}} & \mathbf{0} \\ \hline \mathbf{A}_{\mathrm{i1}} & \mathbf{A}_{\mathrm{i1}} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\mathrm{i2}} & \mathbf{A}_{\mathrm{i2}} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{\mathrm{i3}} & \mathbf{A}_{\mathrm{i3}} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{I} & \mathbf{0} \\ \hline \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{S}_{\mathrm{L}}^{\mathrm{T}} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{S}_{\mathrm{R}}^{\mathrm{T}} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_{\mathrm{L}} & \mathbf{0} & \mathbf{0} & -\mathbf{S}_{\mathrm{R}} & \mathbf{H} \end{array}\right] \cdot \left[\begin{array}{c} \mathbf{v}_{\mathrm{L}} \\ \mathbf{v}_{2} \\ \mathbf{v}_{3} \\ \hline \mathbf{v}_{\mathrm{R}} \\ \mathbf{i}_{\mathrm{L}} \\ \mathbf{i}_{2} \\ \hline \mathbf{i}_{3} \\ \hline -\mathbf{i}_{\mathrm{R}} \\ \hline \mathbf{v}_{\mathrm{N}} \end{array}\right]^{j} = \left[\begin{array}{cccc|cccc|c} -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{\mathrm{v1}} & \mathbf{B}_{\mathrm{v1}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{\mathrm{v2}} & \mathbf{B}_{\mathrm{v2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{\mathrm{v3}} & \mathbf{B}_{\mathrm{v3}} & \mathbf{0} \\ \hline \mathbf{B}_{\mathrm{i1}} & \mathbf{B}_{\mathrm{i1}} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\mathrm{i2}} & \mathbf{B}_{\mathrm{i2}} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{\mathrm{i3}} & \mathbf{B}_{\mathrm{i3}} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F} \end{array}\right] \cdot \left[\begin{array}{c} \mathbf{v}_{\mathrm{L}} \\ \mathbf{v}_{2} \\ \mathbf{v}_{3} \\ \hline \mathbf{v}_{\mathrm{R}} \\ \mathbf{i}_{\mathrm{L}} \\ \mathbf{i}_{2} \\ \hline \mathbf{i}_{3} \\ \hline -\mathbf{i}_{\mathrm{R}} \\ \hline \mathbf{v}_{\mathrm{N}} \end{array}\right]^{j-1} + \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{i}_{\mathrm{N}} \end{array}\right]^{j}$$

Boundary conditions via MNA and implicit Euler methods

Sensitivity determination

- **Sensitivity** with respect to a parameter γ $(j = 1,2,\ldots)$

	Parameter	$\partial\mathbf{A}/\partial\gamma$	$\partial\mathbf{B}/\partial\gamma$
Distributed	$\gamma \in \{\mathbf{L}_0(x),\mathbf{R}_0(x)\}$	$\begin{bmatrix} 0 & \partial\mathbf{A}_v/\partial\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \partial\mathbf{B}_v/\partial\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
	$\gamma \in \{\mathbf{C}_0(x),\mathbf{G}_0(x)\}$	$\begin{bmatrix} 0 & 0 & 0 \\ \partial\mathbf{A}_i/\partial\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ \partial\mathbf{B}_i/\partial\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
	$\gamma \equiv l$	$\begin{bmatrix} 0 & \mathbf{A}_v/l & 0 \\ \mathbf{A}_i/l & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \mathbf{B}_v/l & 0 \\ \mathbf{B}_i/l & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Lumped	$\gamma \in \mathbf{C}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial\mathbf{C}/\partial\gamma \end{bmatrix}\frac{1}{\Delta t}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial\mathbf{C}/\partial\gamma \end{bmatrix}\frac{1}{\Delta t}$
	$\gamma \in \mathbf{G}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial\mathbf{G}/\partial\gamma \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

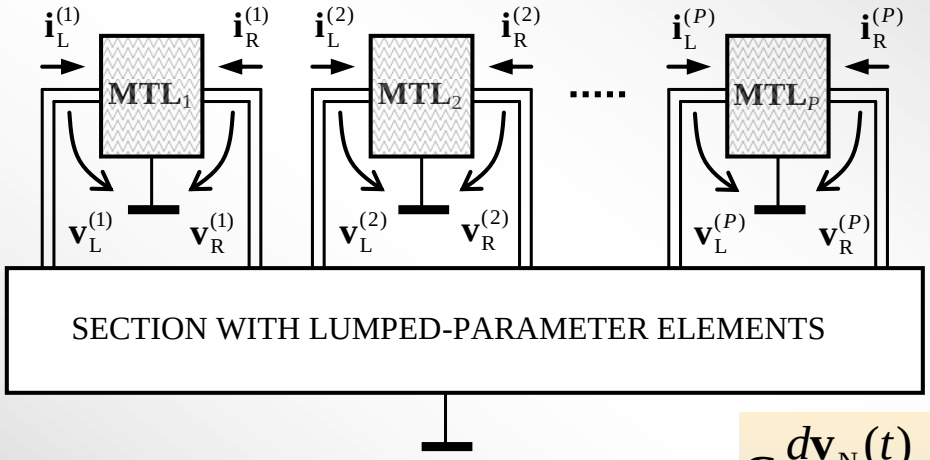
$$\frac{\partial \mathbf{x}^j}{\partial\gamma} = \mathbf{A}^{-1}\left(\mathbf{B}\frac{\partial \mathbf{x}^{j-1}}{\partial\gamma} - \frac{\partial\mathbf{A}}{\partial\gamma}\mathbf{x}^j + \frac{\partial\mathbf{B}}{\partial\gamma}\mathbf{x}^{j-1} + \frac{\partial\mathbf{D}^j}{\partial\gamma}\right)$$

with $\quad \mathbf{x}^j = \mathbf{A}^{-1}\left(\mathbf{B}\mathbf{x}^{j-1} + \mathbf{D}^j\right)$

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_{\pm} & \mathbf{A}_v & \mathbf{0} \\ \mathbf{A}_i & \mathbf{I}_{\pm} & \mathbf{0} \\ \mathbf{I}_2 & \mathbf{0} & \mathbf{S}_c \\ \mathbf{0} & \mathbf{S}_r & \mathbf{H} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{I}_{\mp} & \mathbf{B}_v & \mathbf{0} \\ \mathbf{B}_i & \mathbf{I}_{\mp} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{F} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{0} \\ \mathbf{i}_N \end{bmatrix}$$

Param.	$\partial\mathbf{A}_{vk}/\partial\gamma$	$\partial\mathbf{B}_{vk}/\partial\gamma$	$\partial\mathbf{A}_{ik}/\partial\gamma$	$\partial\mathbf{B}_{ik}/\partial\gamma$
$\gamma \in \mathbf{L}_{0k}$	$-\frac{\partial\mathbf{L}_{0k}}{\partial\gamma}\frac{\Delta x}{\Delta t}$	$-\frac{\partial\mathbf{L}_{0k}}{\partial\gamma}\frac{\Delta x}{\Delta t}$	0	0
$\gamma \in \mathbf{R}_{0k}$	$-\frac{\partial\mathbf{R}_{0k}}{\partial\gamma}\frac{\Delta x}{2}$	$\frac{\partial\mathbf{R}_{0k}}{\partial\gamma}\frac{\Delta x}{2}$	0	0
$\gamma \in \mathbf{C}_{0k}$	0	0	$-\frac{\partial\mathbf{C}_{0k}}{\partial\gamma}\frac{\Delta x}{\Delta t}$	$-\frac{\partial\mathbf{C}_{0k}}{\partial\gamma}\frac{\Delta x}{\Delta t}$
$\gamma \in \mathbf{G}_{0k}$	0	0	$-\frac{\partial\mathbf{G}_{0k}}{\partial\gamma}\frac{\Delta x}{2}$	$\frac{\partial\mathbf{G}_{0k}}{\partial\gamma}\frac{\Delta x}{2}$

General MTL systems (1)



❖ **MTL boundary conditions**

$$\mathbf{v}_{\mathrm{L}}^{(k)}(t)=\mathbf{S}_{\mathrm{L}}^{(k)\mathrm{T}}\mathbf{v}_{\mathrm{N}}(t)\,,\quad \mathbf{v}_{\mathrm{R}}^{(k)}(t)=\mathbf{S}_{\mathrm{R}}^{(k)\mathrm{T}}\mathbf{v}_{\mathrm{N}}(t)$$

❖ **MNA** description

$$\mathbf{C}\frac{d\mathbf{v}_{\mathrm{N}}(t)}{dt}+\mathbf{G}\mathbf{v}_{\mathrm{N}}(t)+\sum_{k=1}^{P}\left(\mathbf{S}_{\mathrm{L}}^{(k)}\mathbf{i}_{\mathrm{L}}^{(k)}(t)+\mathbf{S}_{\mathrm{R}}^{(k)}\mathbf{i}_{\mathrm{R}}^{(k)}(t)\right)=\mathbf{i}_{\mathrm{N}}(t)$$

❖ **Matrix recursive formulation** $\mathbf{A}\mathbf{x}^{j}=\mathbf{B}\mathbf{x}^{j-1}+\mathbf{D}^{j}$

Wendroff method

$$\mathbf{x}^{j}=\left[\mathbf{w}^{(1)\,j\mathrm{T}},\mathbf{w}^{(2)\,j\mathrm{T}},\ldots,\mathbf{w}^{(P)\,j\mathrm{T}},\mathbf{v}_{\mathrm{N}}^{j\mathrm{T}}\right]^{\mathrm{T}}\quad\text{with}\quad\mathbf{w}^{(k)\,j}=\left[\mathbf{v}^{(k)\,j\mathrm{T}},\mathbf{i}^{(k)\,j\mathrm{T}}\right]^{\mathrm{T}}$$

$$\sum_{k=1}^{P}\left(\mathbf{S}_{\mathrm{L}}^{(k)}\mathbf{i}_{\mathrm{L}}^{(k)\,j}+\mathbf{S}_{\mathrm{R}}^{(k)}\mathbf{i}_{\mathrm{R}}^{(k)\,j}\right)+\mathbf{H}\mathbf{v}_{\mathrm{N}}^{j}=\mathbf{F}\mathbf{v}_{\mathrm{N}}^{j-1}+\mathbf{i}_{\mathrm{N}}^{j}$$

$$\mathbf{v}_{\mathrm{L}}^{(k)\,j}-\mathbf{S}_{\mathrm{L}}^{(k)\mathrm{T}}\mathbf{v}_{\mathrm{N}}^{j}=\mathbf{0}\,,\ \mathbf{v}_{\mathrm{R}}^{(k)\,j}-\mathbf{S}_{\mathrm{R}}^{(k)\mathrm{T}}\mathbf{v}_{\mathrm{N}}^{j}=\mathbf{0}$$

MNA equations via implicit Euler method

General MTL systems (2)

❖ **Matrix recursive formulation** $\mathbf{A}\mathbf{x}^{j} = \mathbf{B}\mathbf{x}^{j-1} + \mathbf{D}^{j}$

$$\mathbf{A}^{(k)} = \begin{bmatrix} \mathbf{A}_{\pm}^{(k)} & \mathbf{0} \\ \mathbf{I}_{20}^{(k)} & \mathbf{S}_{\mathrm{c}}^{(k)} \\ \mathbf{S}_{0\mathrm{r}}^{(k)} & \mathbf{H} \end{bmatrix}, \quad \mathbf{B}^{(k)} = \begin{bmatrix} \mathbf{B}_{\mp}^{(k)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c} \mathbf{A}_{\pm}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{\pm}^{(2)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{\pm}^{(P)} & \mathbf{0} \\ \hline \mathbf{I}_{20}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{S}_{\mathrm{c}}^{(1)} \\ \mathbf{0} & \mathbf{I}_{20}^{(2)} & \cdots & \mathbf{0} & \mathbf{S}_{\mathrm{c}}^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_{20}^{(P)} & \mathbf{S}_{\mathrm{c}}^{(P)} \\ \hline \mathbf{S}_{0\mathrm{r}}^{(1)} & \mathbf{S}_{0\mathrm{r}}^{(2)} & \cdots & \mathbf{S}_{0\mathrm{r}}^{(P)} & \mathbf{H} \end{array}\right] \cdot \left[\begin{array}{c} \mathbf{w}^{(1)} \\ \mathbf{w}^{(2)} \\ \vdots \\ \mathbf{w}^{(P)} \\ \hline \mathbf{v}_{\mathrm{N}} \end{array}\right]^{j} = \left[\begin{array}{cccc|c} \mathbf{B}_{\mp}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{\mp}^{(2)} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_{\mp}^{(P)} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{F} \end{array}\right] \cdot \left[\begin{array}{c} \mathbf{w}^{(1)} \\ \mathbf{w}^{(2)} \\ \vdots \\ \mathbf{w}^{(P)} \\ \hline \mathbf{v}_{\mathrm{N}} \end{array}\right]^{j-1} + \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \hline \mathbf{i}_{\mathrm{N}} \end{array}\right]^{j}$$

$$\mathbf{A}_{\pm} = \mathrm{diag}\left(\mathbf{A}_{\pm}^{(1)}, \mathbf{A}_{\pm}^{(2)}, \ldots, \mathbf{A}_{\pm}^{(P)}\right)$$

$$\mathbf{B}_{\mp} = \mathrm{diag}\left(\mathbf{B}_{\mp}^{(1)}, \mathbf{B}_{\mp}^{(2)}, \ldots, \mathbf{B}_{\mp}^{(P)}\right)$$

$$\mathbf{I}_{20} = \mathrm{diag}\left(\mathbf{I}_{20}^{(1)}, \mathbf{I}_{20}^{(2)}, \ldots, \mathbf{I}_{20}^{(P)}\right)$$

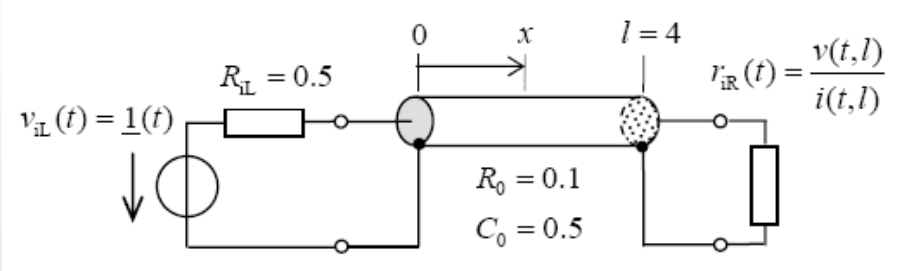
$$\mathbf{S}_{\mathrm{c}} = \left[\mathbf{S}_{\mathrm{c}}^{(1)\mathrm{T}}, \mathbf{S}_{\mathrm{c}}^{(2)\mathrm{T}}, \ldots, \mathbf{S}_{\mathrm{c}}^{(P)\mathrm{T}}\right]^{\mathrm{T}}$$

$$\mathbf{S}_{0\mathrm{r}} = \left[\mathbf{S}_{0\mathrm{r}}^{(1)}, \mathbf{S}_{0\mathrm{r}}^{(2)}, \ldots, \mathbf{S}_{0\mathrm{r}}^{(P)}\right]$$

$$\Rightarrow \begin{bmatrix} \mathbf{A}_{\pm} & \mathbf{0} \\ \mathbf{I}_{20} & \mathbf{S}_{\mathrm{c}} \\ \mathbf{S}_{0\mathrm{r}} & \mathbf{H} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w} \\ \mathbf{v}_{\mathrm{N}} \end{bmatrix}^{j} = \begin{bmatrix} \mathbf{B}_{\mp} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w} \\ \mathbf{v}_{\mathrm{N}} \end{bmatrix}^{j-1} + \begin{bmatrix} \mathbf{0} \\ \mathbf{i}_{\mathrm{N}} \end{bmatrix}^{j}$$

Experimental error analysis

- **Thomson** transmission line (uniform)

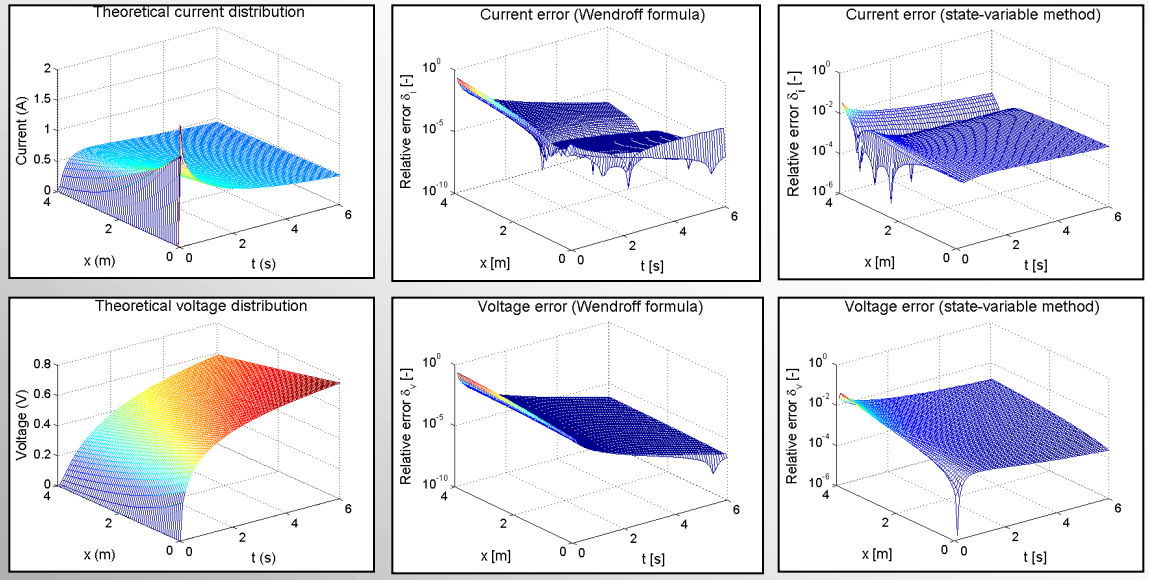


Known analytical solutions

$$i(t,x) = \underline{1}(t)/R_{\mathrm{iL}} \cdot e^{a(t)\left(a(t)+2b(t,x)\right)} \cdot \mathrm{erfc}\left(a(t)+b(t,x)\right)$$

$$v(t,x) = \underline{1}(t) \cdot \mathrm{erfc}\left(b(t,x)\right) - R_{\mathrm{iL}} i(t,x) \ , \ \text{where}$$

$$a(t) = \sqrt{R_0 t / C_0}\,/R_{\mathrm{iL}} \ , \ b(t,x) = x/2 \cdot \sqrt{R_0 C_0 / t}$$



Relative errors for implicit **Wendroff** vs. **state-variable** methods

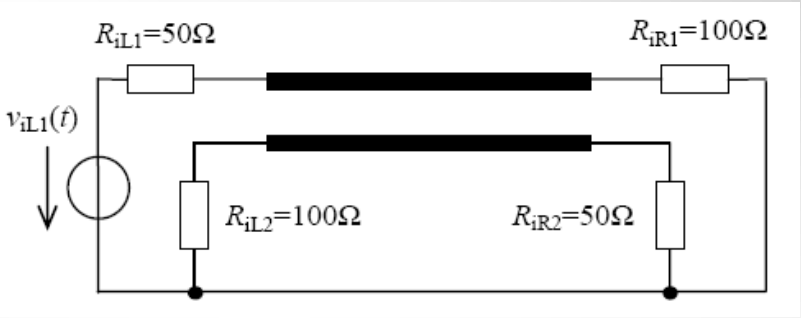
Examples: Thévenin equivalents

- **Uniform/Nonuniform MTLs**: responses to external driving

$$\mathbf{R}_{\mathrm{iL}} = \begin{bmatrix} R_{\mathrm{iL1}} & 0 \\ 0 & R_{\mathrm{iL2}} \end{bmatrix}$$

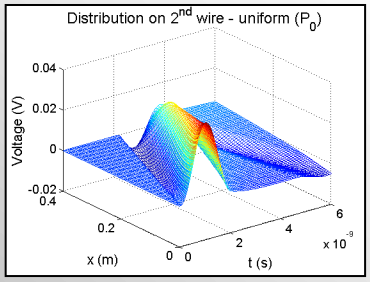
$$\mathbf{v}_{\mathrm{iL}} = \begin{bmatrix} v_{\mathrm{iL1}} \\ 0 \end{bmatrix}$$

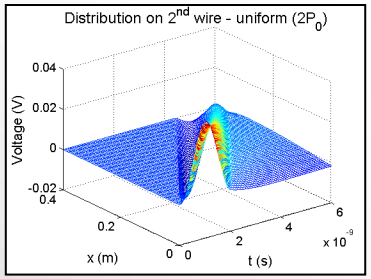
$$\mathrm{v}_{\mathrm{iL1}}(t) = \sin^2(\pi t/2 \cdot 10^{-9})$$

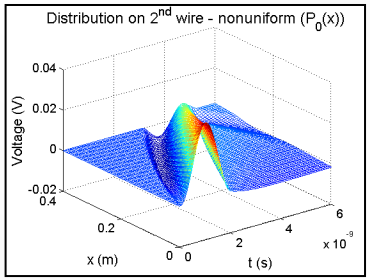


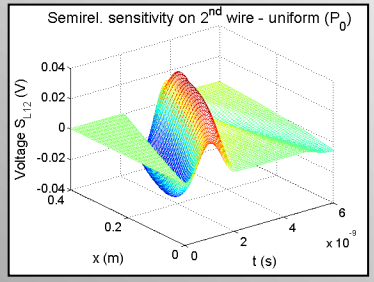
$$\mathbf{R}_{\mathrm{iR}} = \begin{bmatrix} R_{\mathrm{iR1}} & 0 \\ 0 & R_{\mathrm{iR2}} \end{bmatrix}$$

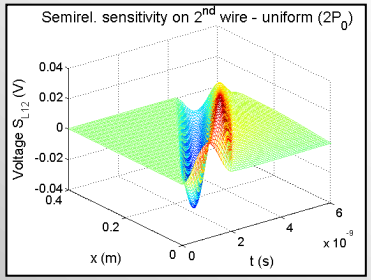
$$\mathbf{v}_{\mathrm{iR}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

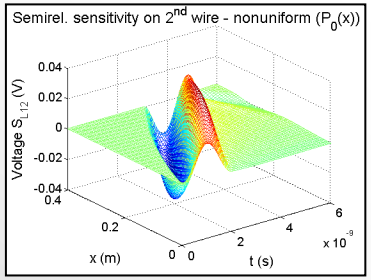












$$\mathbf{P}_0(x) = \mathbf{P}_0 e^{px} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} e^{px}$$

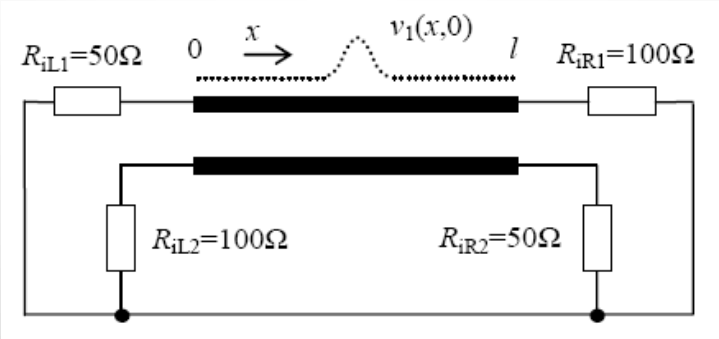
$$\mathbf{P}_0 \in \{\mathbf{R}_0, \mathbf{L}_0, \mathbf{G}_0, \mathbf{C}_0\}$$

Voltage distributions and their sensitivities: uniform vs. nonuniform MTL

Examples: Thévenin equivalents

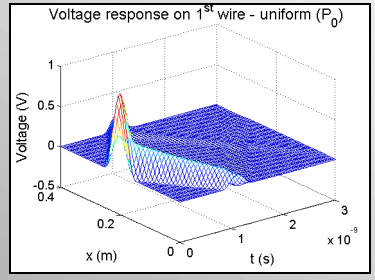
❖ **Uniform/Nonuniform MTLs**:

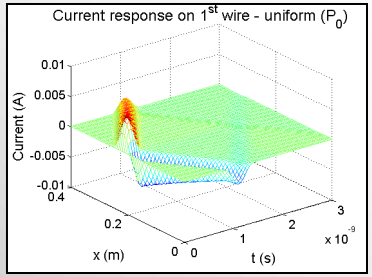
- **nonzero** initial condition



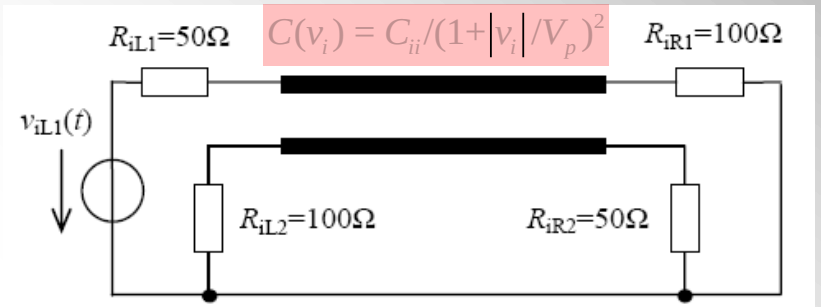
$$v_1(x,0) = \sin^2\left(\pi(4x/l - 3/2)\right), \text{ if } 3l/8 < x < 5l/8$$

$$v_1(x,0) = 0, \text{ otherwise}$$



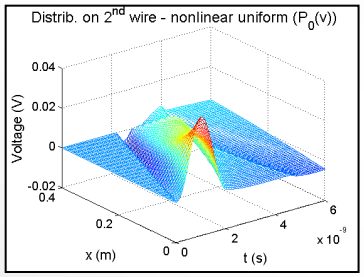


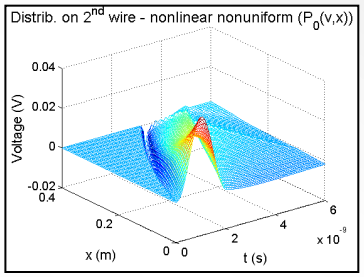
- external driving & **nonlinear** MTL



$$v_{iL1}(t) = \sin^2(\pi t/2\cdot 10^{-9}), \text{ if } 0 \le t \le 2\cdot 10^{-9}$$

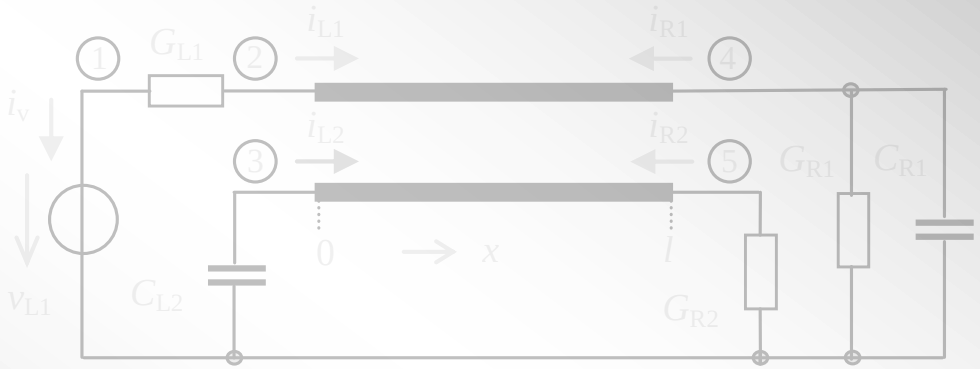
$$v_{iL1}(t) = 0, \text{ otherwise}$$





Example: MNA + Euler method (1)

- **Nonuniform MTL, reactive terminations**

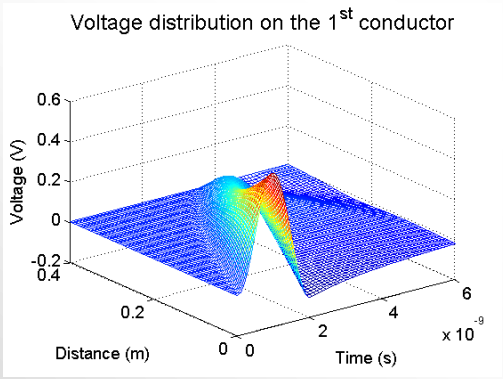


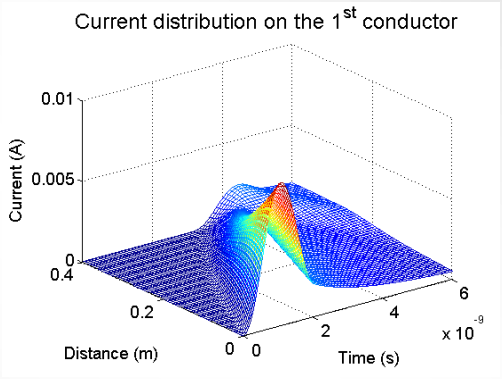
$$\mathbf{C}\frac{d\mathbf{v}_N(t)}{dt}+\mathbf{G}\mathbf{v}_N(t)+\mathbf{S}_L\mathbf{i}_L(t)+\mathbf{S}_R\mathbf{i}_R(t)=\mathbf{i}_N(t)$$

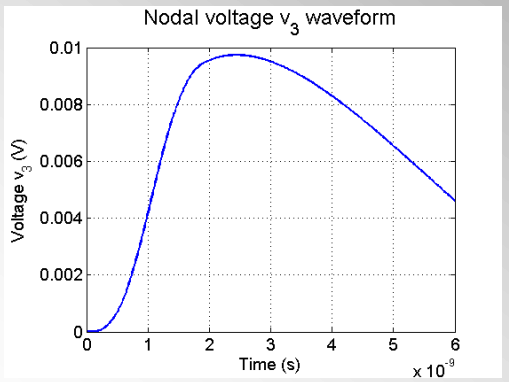
$$\mathbf{G}=\begin{bmatrix} G_{L1} & -G_{L1} & 0 & 0 & 0 & 1 \\ -G_{L1} & G_{L1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{R1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{R2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{S}_L=\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{S}_R=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{C}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{L2} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{R1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{i}_N=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{L1} \end{bmatrix}, \mathbf{i}_L=\begin{bmatrix} i_{L1} \\ i_{L2} \end{bmatrix}, \mathbf{i}_R=\begin{bmatrix} i_{R1} \\ i_{R2} \end{bmatrix}.$$

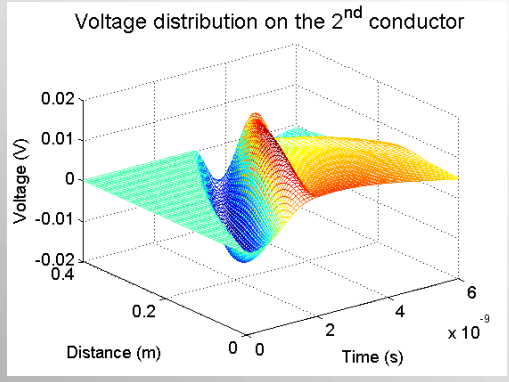
Example: MNA + Euler method (2)

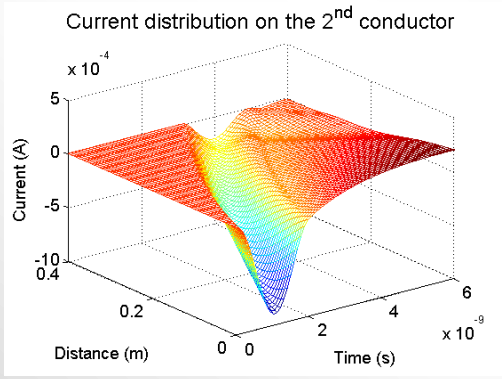
- Voltage and current distributions
- Nodal voltage waveforms

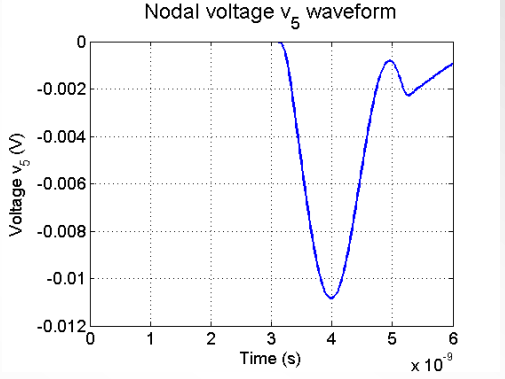






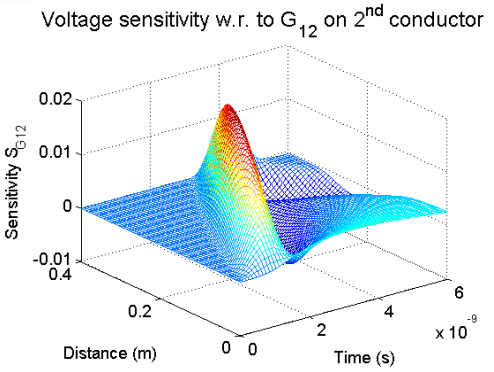


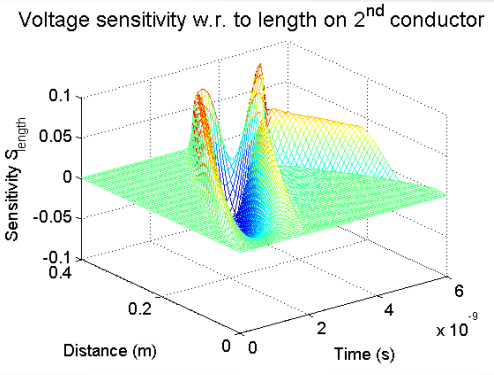


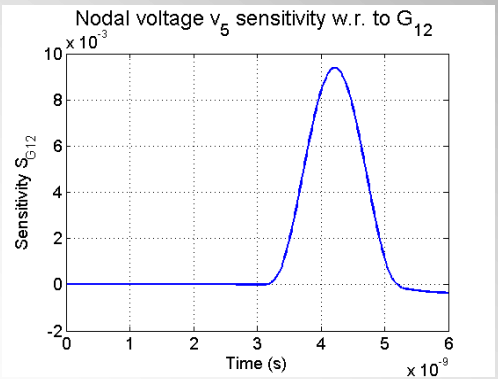


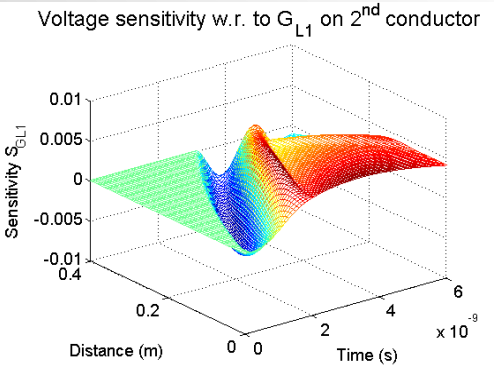
Example: MNA + Euler method (3)

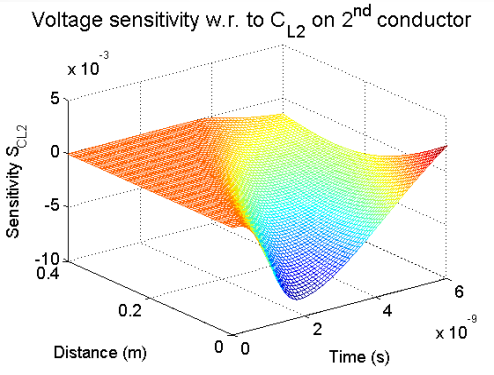
- Voltage distributions sensitivities
- Nodal voltage sensitivities

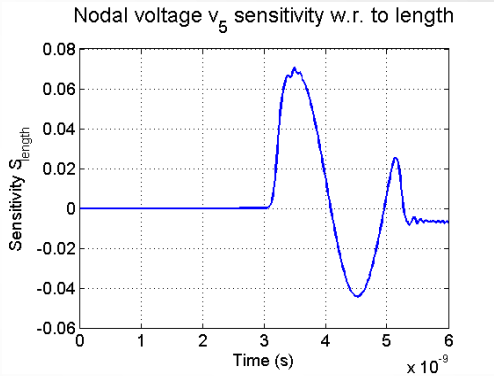




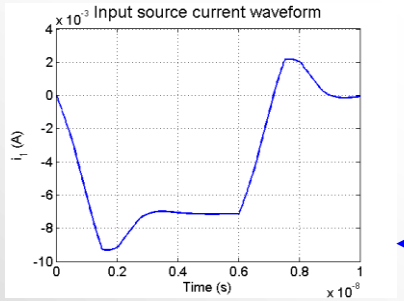


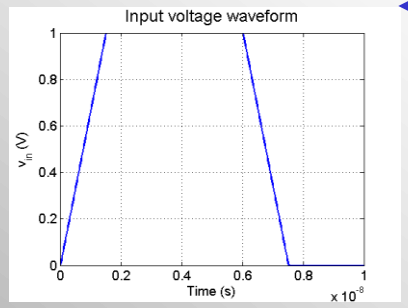


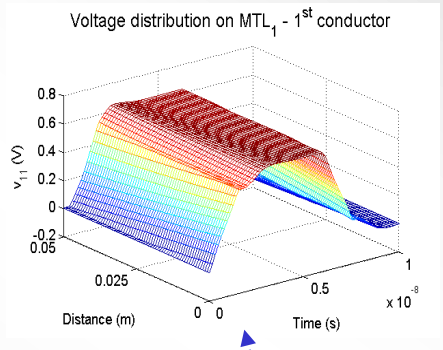


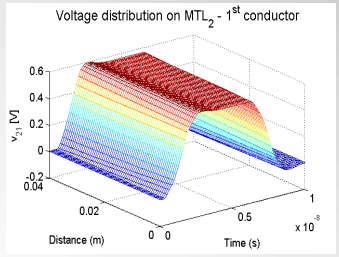


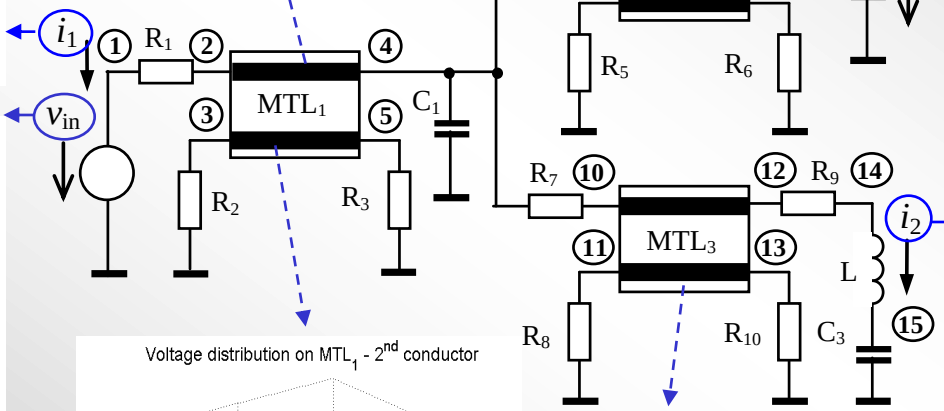
Example: General MTL system

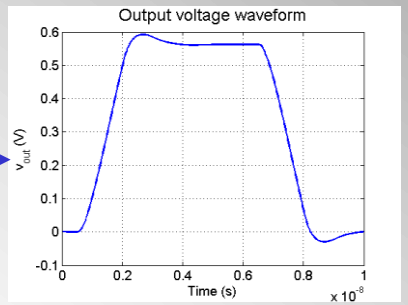


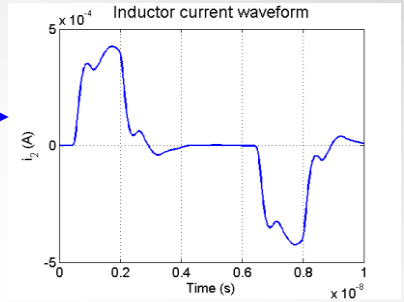


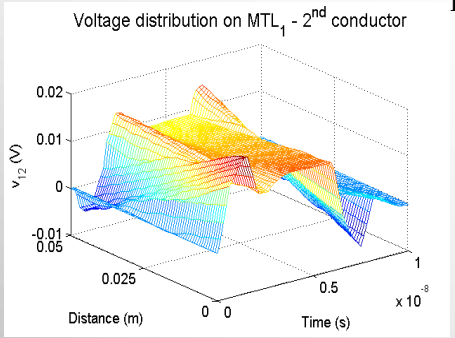


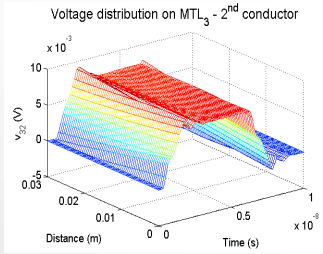






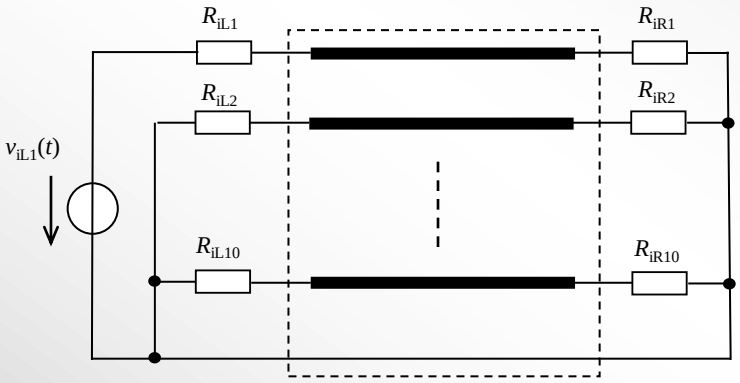


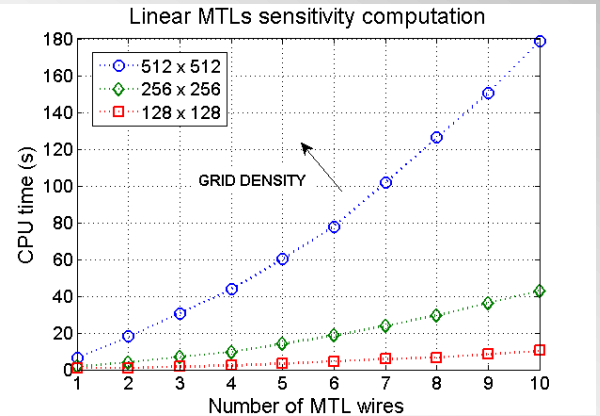


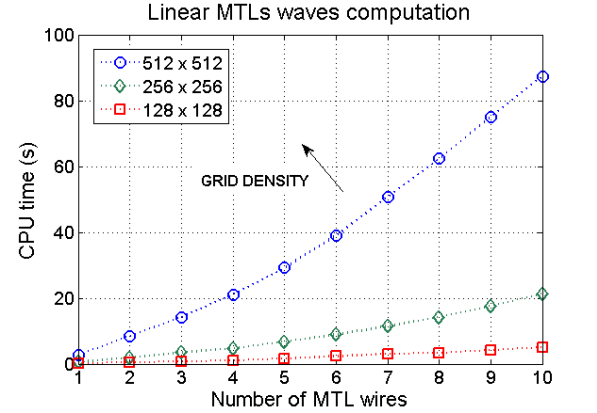


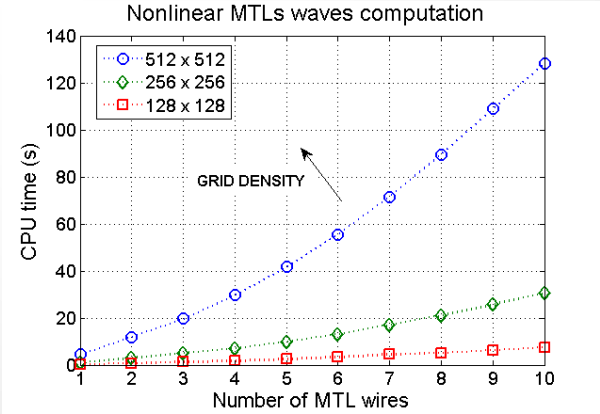
CPU times evaluation

- CPU times for PC 2GHz/2GB, sparse matrix notations









Conclusion

- In case of linear MTLs, the method is stable and accurate enough,
- In case of nonlinear MTLs, further studies will be performed,
- Higher-order techniques will be investigated in future as well

Thank you for your kind attention !

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